

# QM treatment

The trap potential may be written as :

$$\hat{V}(t) = \frac{m}{2} W(t) \hat{x}^2 \quad \text{where} \quad W(t) = \frac{\omega_{rf}^2}{4} [a_x + 2q_x \cos(\omega_{rf} t)]$$

With these definition the Hamiltonian looks :

$$\hat{H}^m = \frac{\hat{p}^2}{2m} + \frac{m}{2} W(t) \hat{x}^2$$

Reminder:  $u(\zeta) = A e^{\{i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{i2n\zeta\}} + B e^{\{-i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{-i2n\zeta\}}$



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$$\hat{H}^m = \frac{\hat{p}^2}{2m} + \frac{m}{2} W(t) \hat{x}^2$$

The equation of motion of the operators in Heisenberg picture are:

$$\dot{\hat{x}} = \frac{1}{i\hbar} [\hat{x}, \hat{H}^m] = \frac{\hat{p}}{m} \quad \dot{\hat{p}} = \frac{1}{i\hbar} [\hat{p}, \hat{H}^m] = -mW(t)\hat{x}$$

By combining we obtain:

$$\ddot{\hat{x}} + W(t)\hat{x} = 0$$

This is equivalent to Mathieu equation (not surprising!!) provided  $\hat{x}$ \_is replace by u(t) function. So to solve this Hamiltonian, we use the special solution of Mathieu equation subject to boundary conditions

Reminder:  $u(\zeta) = A e^{\{i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{i2n\zeta\}} + B e^{\{-i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{-i2n\zeta\}}$



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Reminder:  $u(\zeta) = Ae^{\{i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{i2n\zeta\}} + Be^{\{-i\beta_x \zeta\}} \sum_{n=-\infty}^{\infty} C_{2n} e^{\{-i2n\zeta\}}$

$$u(0) = 1, \quad \dot{u}(0) = i\nu$$

These boundary condition implies  $A = 1, B = 0$

$$u(t) = e^{\frac{i\beta_x \omega_{rf} t}{2}} \sum_{n=-\infty}^{\infty} C_{2n} e^{in\omega_{rf} t} = e^{\frac{i\beta_x \omega_{rf} t}{2}} \Phi(t)$$

Periodic with period  $T = \frac{2\pi}{\omega_{rf}}$

Therefore the coefficients takes the form:

$$\sum_{n=-\infty}^{\infty} C_{2n} = 1$$

$$u(0) = 1$$

$$\nu = \omega_{rf} \sum_{n=-\infty}^{\infty} C_{2n} \left( \frac{\beta_x}{2} + n \right)$$

$$\dot{u}(0) = i\nu$$

This solution and its complex conjugate are linearly independent and hence they obey Worskian identity



# QM treatment

This solution and its complex conjugate are linearly independent and hence they obey Wronskian identity

$$u^*(t)\dot{u}(t) - u(t)\dot{u}^*(t) = u^*(0)\dot{u}(0) - u(0)\dot{u}^*(0) = 2i\nu$$

Similar argument holds for  $\hat{x}(t)$  and  $u(t)$  as both obey the same differential equations, so a complex linear combination as

$$\hat{C}(t) = \sqrt{\frac{m}{2\hbar\nu}} i \{u(t)\dot{\hat{x}}(t) - \dot{u}(t)\hat{x}(t)\}$$

Is also proportional to their Wronskian identity and also constant in time



# QM treatment

$$\hat{C}(t) = \hat{C}(0) = \sqrt{\frac{1}{2m\hbar\nu}} \{mv\hat{x}(0) + i\hat{p}(0)\}$$

This is familiar annihilation operator of static HO of mass  $m$  and frequency  $\nu$

$$\hat{C}(t) = \hat{C}(0) = \hat{a} \quad \text{Implies} \quad [\hat{C}, \hat{C}^T] = [\hat{a}, \hat{a}^T] = 1$$

This oscillator which is time independent is known as the reference oscillator

$$\begin{aligned}\hat{x}(t) &= \sqrt{\frac{\hbar}{2m\nu}} \{\hat{a}u^*(t) + \hat{a}^T u(t)\} \\ \hat{p}(t) &= \sqrt{\frac{\hbar m}{2\nu}} \{\hat{a}\dot{u}^*(t) + \hat{a}^T \dot{u}(t)\}\end{aligned}$$



# MAKING OF A QUBIT

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## Content

1. Light matter interaction
2. Cooling of ions
3. Single qubit operations
4. Multi-qubit operations

# Light matter interaction

Time dependent SE

Stationary states of atom

$$H_0|\phi_k\rangle = E_k|\phi_k\rangle$$

Any state in the atomic basis

$$|\psi\rangle = \sum_k c_k |\phi_k\rangle$$

Plugging back to TDSE

$$i\hbar \frac{\partial}{\partial t} \sum_k c_k |\phi_k\rangle = [H_0 + H'(t)] \sum_k c_k |\phi_k\rangle$$

Multiplying both sides by  $\langle\phi_j|$  on both sides

$$i\hbar \frac{\partial}{\partial t} c_j(t) = \sum_k H'_{jk} c_k(t) e^{i\omega_{jk}t}$$

$$i\hbar \frac{\partial |\psi(\vec{r}, t)\rangle}{\partial t} = H(t)|\psi(t)\rangle$$

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = [H_0 + H'(t)]|\psi\rangle$$

atom

Light-atom

$$H'_{jk} = \langle\phi_j|H'(t)|\phi_k\rangle$$

$$\omega_{jk} = (\omega_j - \omega_k)$$



$$i\hbar \frac{\partial}{\partial t} c_j(t) = \sum_k H'_{jk} c_k(t) e^{i\omega_{jk}t}$$

This equation is exact but not possible to solve without approximating

We are interested in laser light interacting with an atom. Therefore assuming the laser to be of single frequency and addressing only two states of the atom. Therefore truncate the summation to only two states:

Two-level system interacting with light:

$$i\hbar \frac{dc_g(t)}{dt} = c_e(t) H'_{ge}(t) e^{-i\omega_a t}$$

$$i\hbar \frac{dc_e(t)}{dt} = c_g(t) H'_{eg}(t) e^{i\omega_a t}$$

$j = g; k = e$  ground and excited state  
 $\omega_a = \omega_e - \omega_g$  atomic resonance frequency



Now we need to calculate the exact form of  $H'_{ge}(t)$  for light matter interaction  
(2-level approximation)

$$H = \frac{p^2}{2m} + V(r)$$

KE                          Coulomb energy

Reminder (EM-II):

$$\vec{A}(\vec{r}, t) = (A_0 \hat{\epsilon}_z e^{i(ky - \omega t)} + A_0^* \hat{\epsilon}_z e^{-i(ky - \omega t)})$$

$$\frac{E}{2} = -\frac{\partial A}{\partial t} = i\omega A_0$$

$$\frac{B}{2} = \nabla \times A = ikA_0$$

$$H = \frac{(P - eA)^2}{2m} + V(r) - \frac{e}{m} \vec{S} \cdot \vec{B}$$

$$= \frac{P^2}{2m} + V(r) + \frac{e^2 A^2}{2m} - \frac{e}{2m} (\vec{P} \cdot \vec{A} + \vec{A} \cdot \vec{P}) - \frac{e}{m} \vec{S} \cdot \vec{B}$$

Energy of EM field                          E field - charge interaction                          B field spin interaction



$$H = \frac{(P - eA)^2}{2m} + V(r) - \frac{e}{m} \vec{S} \cdot \vec{B}$$

$$= \frac{P^2}{2m} + V(r) + \cancel{\frac{e^2 A^2}{2m}} - \frac{e}{2m} (\vec{P} \cdot \vec{A} + \vec{A} \cdot \vec{P})$$

~~$e^2 A^2 / 2m$~~

~~$\frac{e}{m} \vec{S} \cdot \vec{B}$~~

Neglect  $A^2$  as compared to  $A$

Neglect  $\vec{S} \cdot \vec{B}$  as compared to  $\vec{P} \cdot \vec{A}$

$$= H_0 - \frac{e}{m} \vec{P} \cdot \vec{A}$$

$$= H_0 - \frac{e}{m} P_z [A_0 e^{iky} e^{-i\omega t} - A_0^* e^{-iky} e^{i\omega t}]$$

Dipole approximation only 1<sup>st</sup> term is kept

$$= H_0 - \frac{eE}{m\omega} P_z \sin \omega t$$

Expanding the exponential factor

$$e^{\pm iky} = e^{\pm \frac{i2\pi y}{\lambda}} = 1 \pm iky - \frac{k^2 y^2}{2} \dots$$

By proper choice of gauge it can be shown to be equivalent to

$$= H_0 - e \vec{E} \cdot \vec{r}$$

$$= H_0 + H_I$$

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Two most important interactions are electric dipole and electric quadrupole

$$H_D = e \vec{E} \cdot \vec{r}$$

$$H_Q = e Q_{ij} \frac{\partial E_i}{\partial x_j} = \frac{1}{2} e \left( x_i x_j - \frac{1}{3} \delta_{ij} x^2 \right) = \frac{1}{2} e k z (\vec{E} \cdot \vec{z}) [ie^{i(ky-\omega t)} + c.c.]$$

In matrix representation in the basis of  $|e\rangle$  and  $|g\rangle$

$$H_{D/Q} = \hbar \Omega_0^{D/Q} (|g\rangle\langle e| + |e\rangle\langle g|) \times [e^{i(kx-\omega t+\phi)} + e^{-i(kx-\omega t+\phi)}].$$

Where,

$$\frac{\hbar}{2} \Omega_0^D = e \langle g | \vec{E} \cdot \vec{r} | e \rangle$$

Dipole transition

$$\frac{\hbar}{2} \Omega_0^Q = \frac{ek}{2} \langle g | |\vec{r}| (\vec{E} \cdot \vec{r}) | e \rangle$$

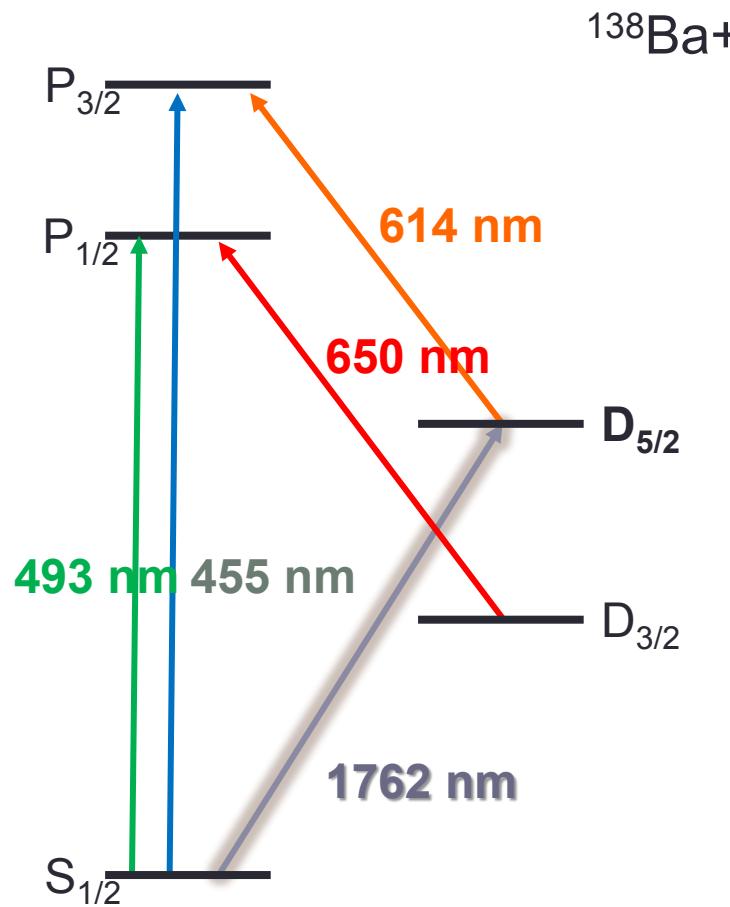
Quadrupole transition

$$\frac{\hbar}{2} \Omega_0^{RT} = -\hbar \frac{|\Omega_{g3} \Omega_{e3}|}{\Delta_R} e^{i\Delta\phi}$$

Two photon Raman transition

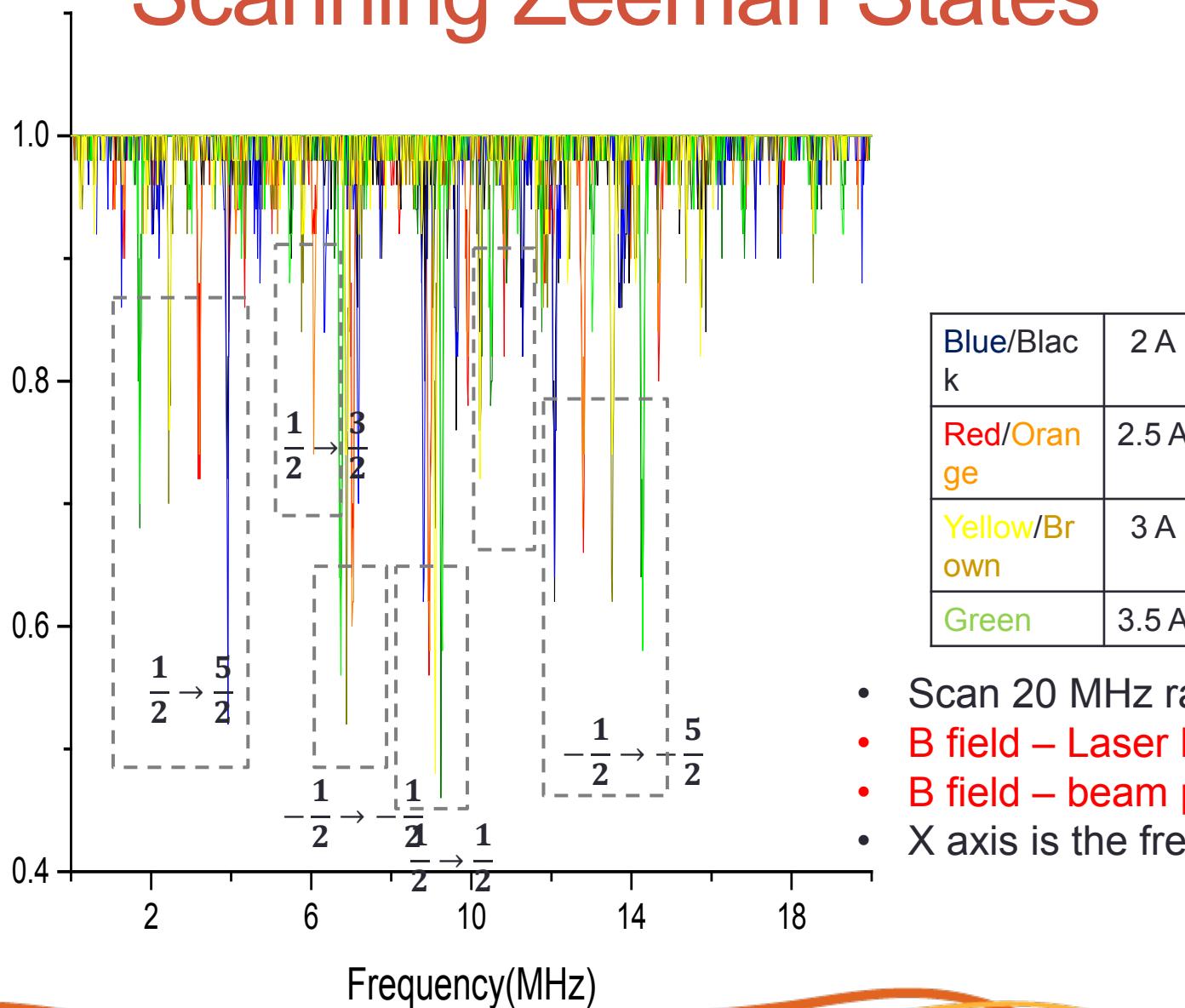


# 138Ba+ Ion Energy Level @ D<sub>5/2</sub>



- **$S_{1/2}$  -  $D_{5/2}$  transition**  
 $1762.1745 \text{ nm (170.12643 THz)}$   
 $T : 34.5 \text{ sec.}$

# Scanning Zeeman States



# Total Hamiltonian: trapped 2-level ion

$$\begin{aligned} H &= H_{atom} + H_{trap} + H_I \\ &= H_0 + H_I \end{aligned}$$

Atomic Hamiltonian:

$$H_{atom} = \frac{\hbar}{2} \omega_a (|e\rangle\langle e| - |g\rangle\langle g|) = \frac{\hbar}{2} \omega_a \sigma_z$$

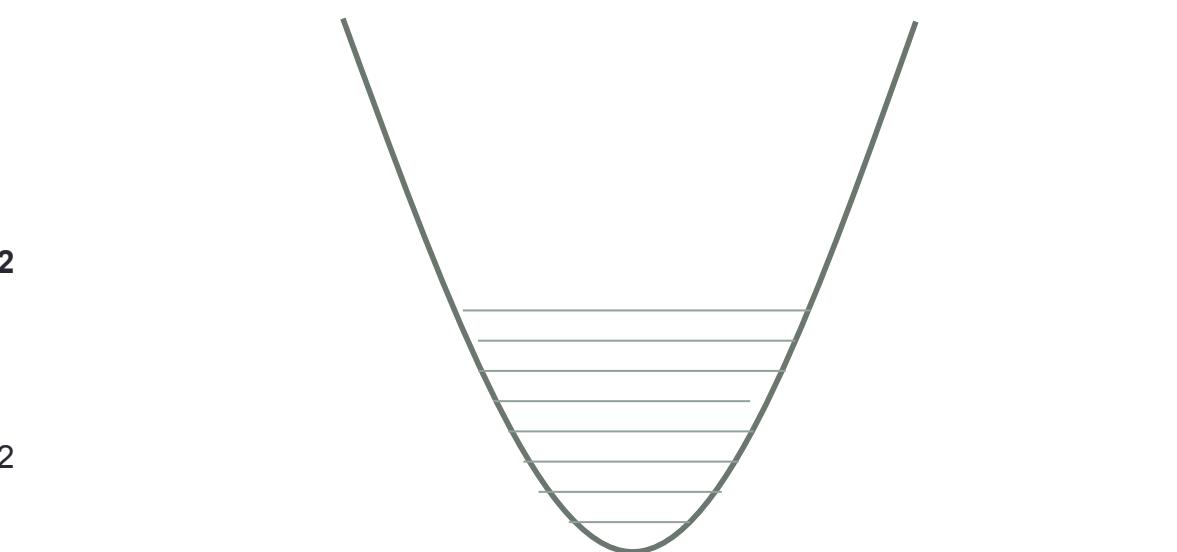
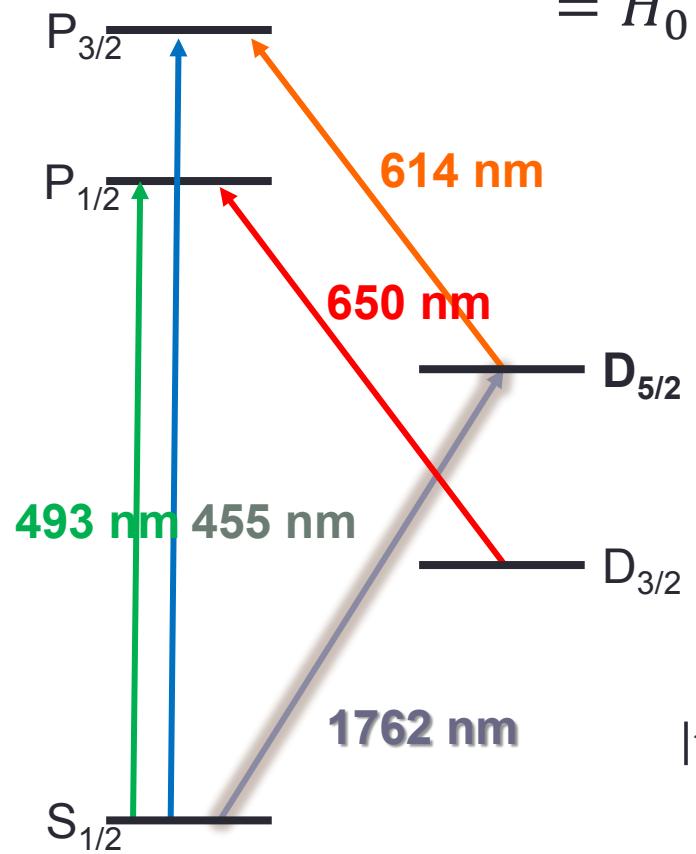
$$H_{trap} = \frac{\hat{p}^2}{2m} + \frac{m}{2} \left( \frac{\Omega_{rf}^2}{4} (a_x + 2q_x \cos(\Omega_{rf}t)) \right) \hat{x}^2$$

The solution of the unperturbed Hamiltonian is completely known, therefore



# Total Hamiltonian: trapped 2-level ion

$$H = H_{atom} + H_{trap} + H_I \\ = H_0 + H_I$$

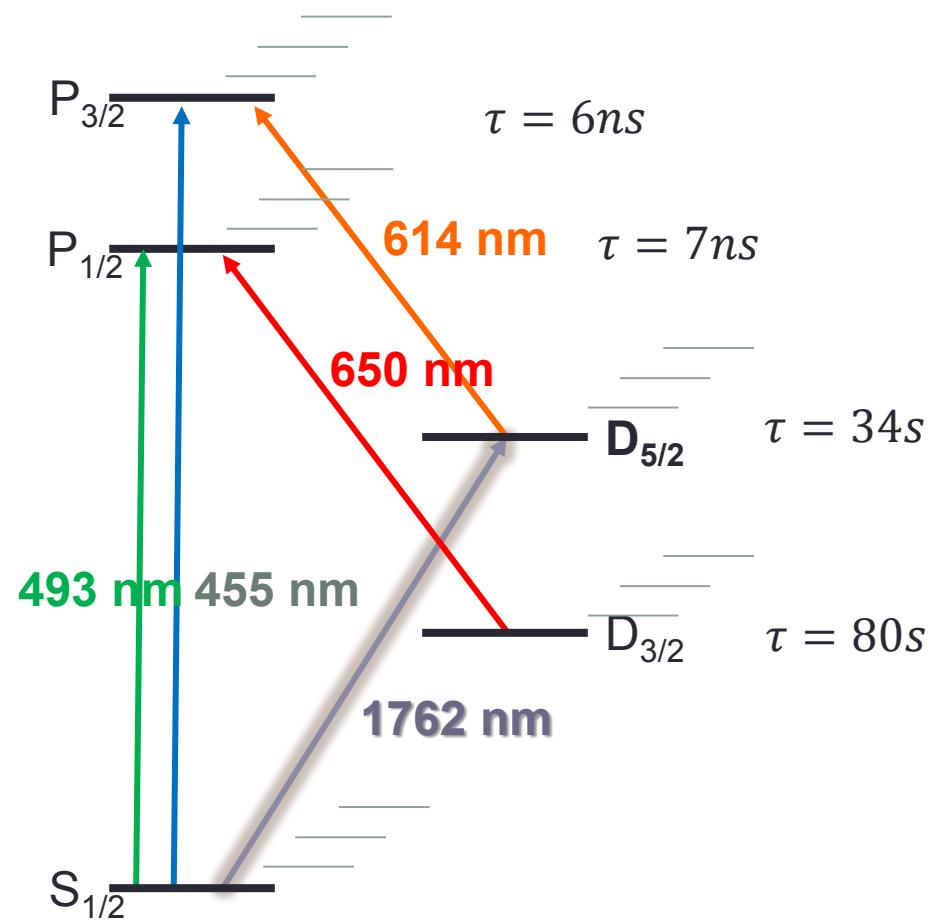
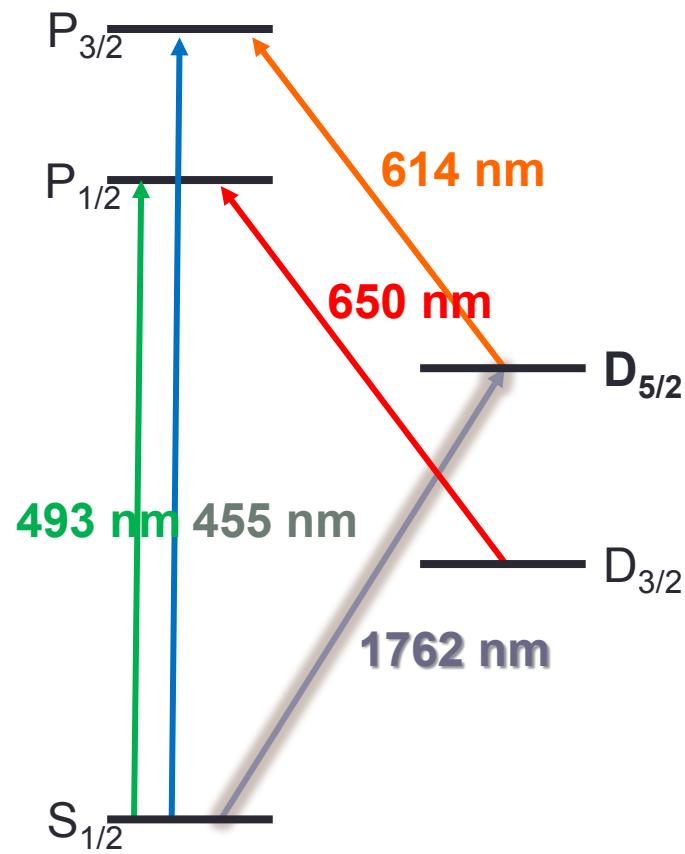


$$|\psi\rangle_{sys} = |\psi\rangle_{at} \times |\psi\rangle_{trap}$$



# Total Hamiltonian: trapped 2-level ion

## The concept



$H_I$  in interaction picture is:

$$H_{inter} = U_0^\dagger H_I U_0$$

$$\text{Reminder } U_0 = e^{-\frac{i}{\hbar} \widehat{H}_0 t}$$

$$= \frac{\hbar}{2} \Omega e^{\frac{i}{\hbar} H_{atom} t} (\sigma_+ + \sigma_-) e^{-\frac{i}{\hbar} H_{atom} t} \times e^{\frac{i}{\hbar} H_{trap} t} [ e^{i(k\hat{x} - \omega t + \phi)} + e^{-i(k\hat{x} - \omega t + \phi)} ] e^{-\frac{i}{\hbar} H_{trap} t}$$

Baker–Campbell–Hausdorff formula

$$\text{Reminder: } e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \dots$$

$$= \frac{\hbar}{2} \Omega (\sigma_+ e^{i\omega_a t} + \sigma_- e^{-i\omega_a t}) \times e^{\frac{i}{\hbar} H_{trap} t} [ e^{i(k\hat{x} - \omega t + \phi)} + e^{-i(k\hat{x} - \omega t + \phi)} ] e^{-\frac{i}{\hbar} H_{trap} t}$$



- Rotating wave approximation for  $(\omega_a \pm \omega)$
- The transformation of  $H_{trap}$  to interaction picture is same as converting to Heisenberg picture meaning  $\hat{x} \rightarrow \hat{x}(t)$

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\nu_{sec}}} (\hat{a}u^*(t) + \hat{a}^\dagger u(t))$$

Therefore we obtain:

$$H_{inter} = \frac{\hbar}{2} \Omega \sigma_+ e^{i(\phi + \eta[\hat{a}u^*(t) + \hat{a}^\dagger u(t)] - \delta t)} + h.c.$$

Lamb-Dicke parameter;

$$\eta = k \sqrt{\frac{\hbar}{2m\nu_{sec}}}$$



$$H_{inter} = \frac{\hbar}{2} \Omega \sigma_+ e^{i(\phi + \eta[\hat{a}u^*(t) + \hat{a}^\dagger u(t)] - \delta t)} + h.c.$$

Further simplification can be done by considering the parameter regime in which a linear trap works:

$$\begin{aligned} (|a_x|, q_x^2) \ll 1 &\equiv \beta_x \omega_{rf} \sim \nu \\ C_0 \sim \left(1 + \frac{q_x}{2}\right)^{-1} \end{aligned}$$

$$H_{inter}(t) = \frac{\frac{\hbar}{2} \Omega}{1 + \frac{q_x}{2}} \sigma_+ e^{i\eta(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})} e^{i(\phi - \delta t)} + h.c.$$

$$= \frac{\hbar}{2} \Omega_0 \sigma_+ e^{i\eta(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})} e^{i(\phi - \delta t)} + h.c.$$

Special case: Lamb-Dicke regime

$$\eta = k \sqrt{\frac{\hbar}{2m\nu_{sec}}} = \frac{2\pi}{\lambda} \sqrt{\frac{\hbar}{2m\nu_{sec}}}$$

Spread of the wave packet  $\sim 10\text{nm}$

Wavelength of probe light  $\sim 500\text{ nm}$

$$\eta \ll 1$$

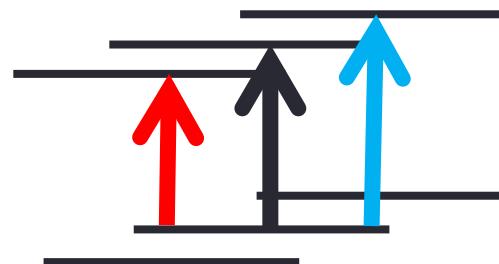
$$H_{inter}(t) = \frac{\hbar}{2} \Omega_0 \sigma_+ \left( 1 + i\eta (\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t}) \right) e^{i(\phi - \delta t)} + h.c.$$



Three cases of importance:

Carrier ( $\delta = 0$ ):

$$H_c = \frac{\hbar}{2} \Omega_0 (\sigma_+ e^{i\phi} + \sigma_- e^{-i\phi})$$



Red Side Band (RSB) ( $\delta = -\nu$ ):

$$H_c = \frac{\hbar}{2} \Omega_0 \eta (\hat{a} \sigma_+ e^{i\phi} + \hat{a}^\dagger \sigma_- e^{-i\phi})$$

$$\Omega_{n,n-1} = \Omega_0 \sqrt{n} \eta$$

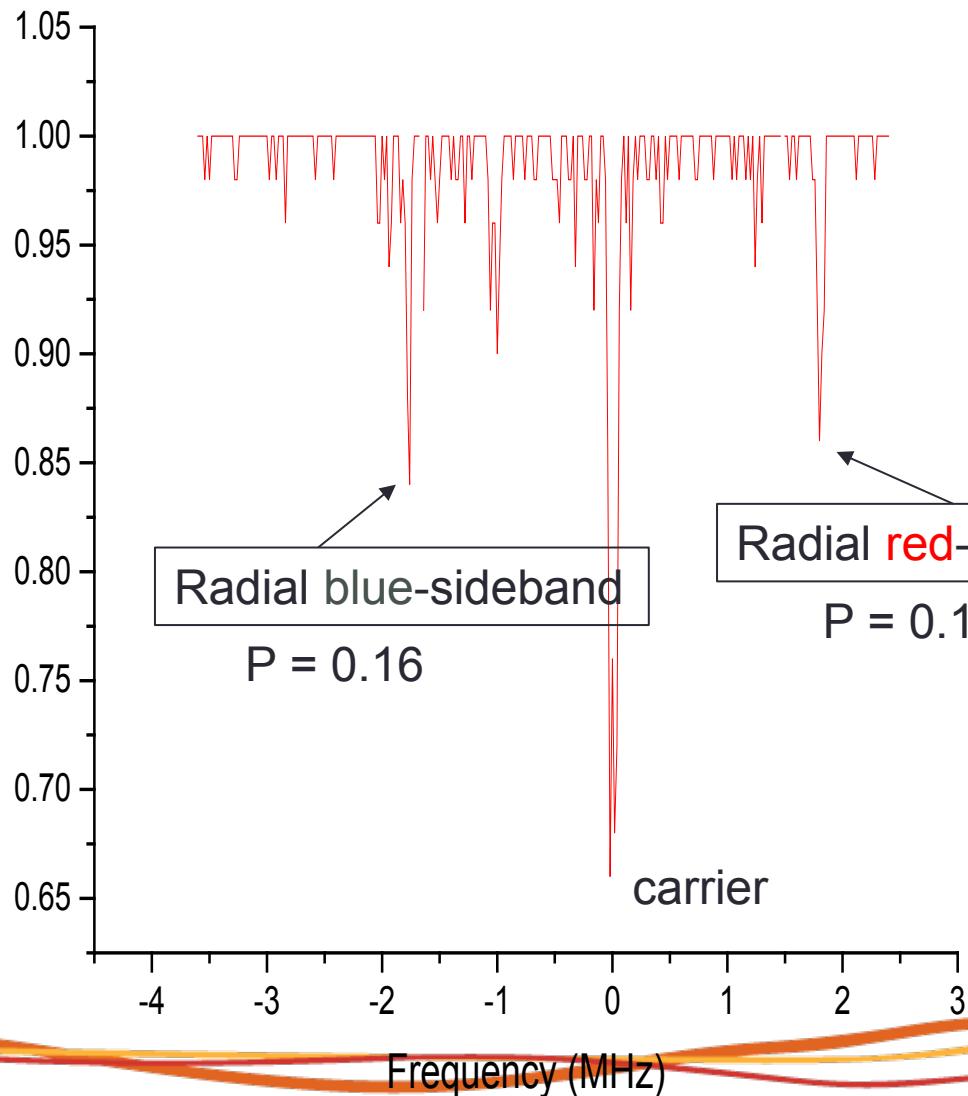
Blue Side Band (BSB) ( $\delta = +\nu$ ):

$$H_c = \frac{\hbar}{2} \Omega_0 \eta (\hat{a}^\dagger \sigma_+ e^{i\phi} + \hat{a} \sigma_- e^{-i\phi})$$

$$\Omega_{n,n+1} = \Omega_0 \sqrt{n+1} \eta$$



# Carrier and Side-band



Power of rf = 4 W  
(trap frequency  $\sim 1.8$  MHz)

$$\frac{P_{\text{red}}}{P_{\text{blue}}} = \frac{n}{n+1} = \frac{0.14}{0.16}$$

$n = 7$

Higher order sidebands  $\delta = l\nu$  with  $|l| \geq 1$

These involve two-”phonons”

$$H_c = \frac{\hbar}{2} \Omega_0 \frac{\eta^2}{2} (\hat{a}^2 \sigma_+ e^{i\phi} + \hat{a}^{\dagger 2} \sigma_- e^{-i\phi})$$

For  $l = -2$ , since it depends on  $\eta^2$  the coupling strength is very low



# Resolved sideband

$$\delta = l\nu + \delta' \text{ where } \delta' \ll \nu$$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_{n,g}(t)|n, g\rangle + c_{n,e}(t)|n, e\rangle$$

$$i\hbar\partial_t|\psi(t)\rangle = \hat{H}_{int}|\psi(t)\rangle \quad [time\ dependent\ SE]$$

$$\begin{aligned}\dot{c}_{n,g} &= -i^{1-|l|} e^{i(\delta't-\phi)} \left( \frac{\Omega_{n+l,n}}{2} \right) c_{n+l,e} \\ \dot{c}_{n+l,e} &= -i^{1+|l|} e^{-i(\delta't-\phi)} \left( \frac{\Omega_{n+l,n}}{2} \right) c_{n,g}\end{aligned}$$

Laplace transform to solve:

$$\begin{bmatrix} c_{(n+l,e)}(t) \\ c_{(n,g)}(t) \end{bmatrix} = T_n^l \begin{bmatrix} c_{n+l,e}(0) \\ c_{n,g}(0) \end{bmatrix}$$

Solutions to TDSE



# Resolved sideband

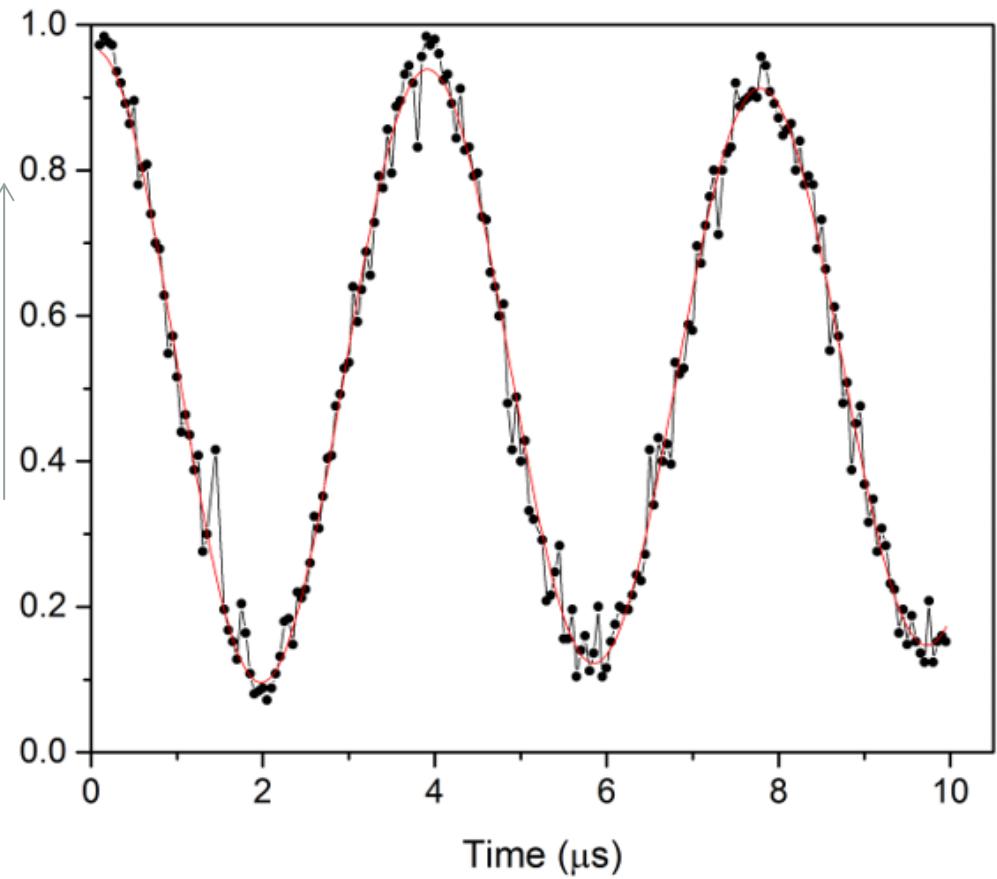
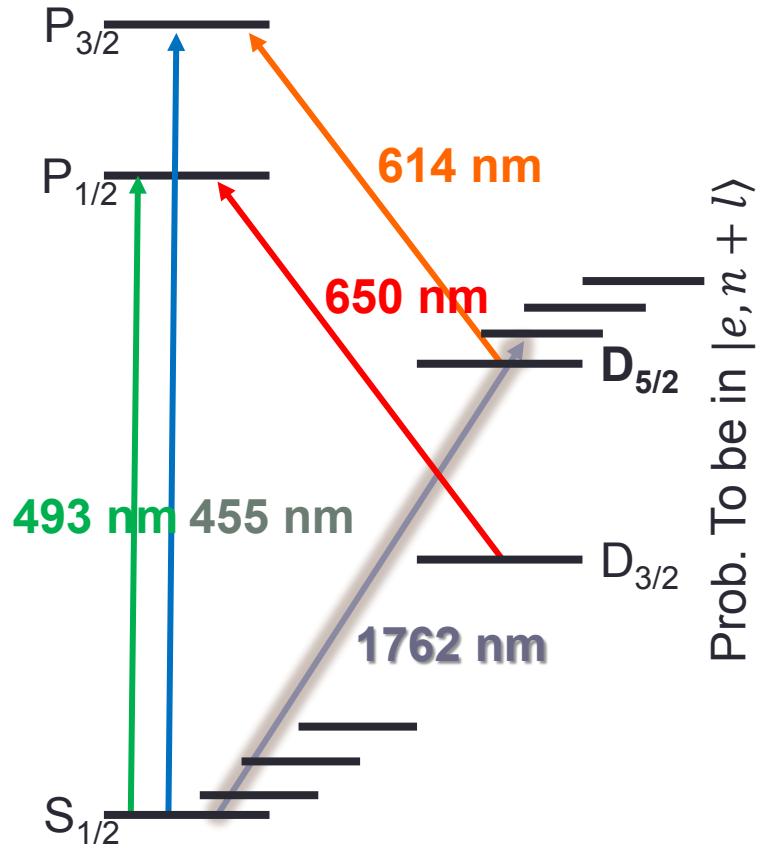
$$T_n^l = \begin{cases} e^{-i\left(\frac{\delta'}{2}\right)t} \left[ \cos\left(\frac{f'_n t}{2}\right) + \frac{i\delta'}{f'_n} \sin\left(\frac{f'_n t}{2}\right) \right] & -i\frac{\Omega_{n+l,n}}{f_n^l} e^{i\left(\phi + \frac{|l|\pi}{2} - \frac{\delta' t}{2}\right)} \sin\left(\frac{f'_n t}{2}\right) \\ -i\frac{\Omega_{n+l,n}}{f_n^l} e^{-i\left(\phi + \frac{|l|\pi}{2} - \frac{\delta' t}{2}\right)} \sin\left(\frac{f'_n t}{2}\right) & e^{i\left(\frac{\delta'}{2}\right)t} \left[ \cos\left(\frac{f'_n t}{2}\right) - \frac{i\delta'}{f'_n} \sin\left(\frac{f'_n t}{2}\right) \right] \end{cases}$$

$$f'_n = \sqrt{\delta'^2 + \Omega_{n+l,n}^2}$$

- Rabi oscillation between the  $|n, g\rangle$  and  $|e, n + l\rangle$
- Side-band cooling / ground state cooling
- Single qubit operation



# Resolved sideband



# Un-resolved sideband

Master equation for 2-level atom in equilibrium with thermal reservoir (master eq. with spontaneous emission given by Liouvillian) :

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [\hat{H}_{trap} + \hat{H}_{atom} + \hat{H}_I, \rho] + L^d \rho$$

where,

$$L^d \rho = \frac{\Gamma}{2} (2\sigma^- \rho' \sigma^+ - \sigma^+ \sigma^- \rho - \rho \sigma^+ \sigma^-)$$

where,

$$\rho' = \frac{1}{2} \int_{-1}^1 dz Y(z) e^{ik\hat{x}z} \rho e^{-ik\hat{x}z} \text{ where, } Y(z) = \frac{3(1+z^2)}{4}$$

- Laser Doppler cooling
- Electro-magnetically induced transparency

# Doppler cooling

$$V_p(x) = \frac{1}{2} m v^2 x^2$$

$$v(t) = v_0 \cos(\nu t)$$

$$\left( \frac{dP}{dt} \right)_{av} = F_{av} = \hbar k \Gamma \rho_{ee}$$

$$\rho_{ee} = \frac{\frac{s}{2}}{1 + s + \left( \frac{2\delta_{eff}}{\Gamma} \right)^2}$$

$$s = \frac{2|\Omega|^2}{\Gamma^2}$$

$$\delta_{eff} = (\omega_l - \omega_{at}) - \vec{k} \cdot \vec{v}$$

Pseudo potential (day-l)

Classical velocity

Rate of change of momentum

excited state population

saturation parameter

effective detuning



# Doppler cooling – the drag

$$F_{av} = \hbar k \Gamma \rho_{ee}$$

Rate of change of momentum

$$F_{av} = F_0(1 + \kappa v)$$

linearizing force in terms of velocity

where

$$F_0 = \frac{\hbar k \Gamma \frac{s}{2}}{1+s+\left(\frac{2\Delta}{\Gamma}\right)^2}$$

definition of force that displaces the ion

$$\kappa = \frac{\frac{8k\Delta}{\Gamma^2}}{1+s+\left(\frac{2\Delta}{\Gamma}\right)^2}$$

definition of the friction

So we have generated a viscous drag force provided  $\Delta < 0$

# Doppler cooling – rate and final temp.

$$\dot{E}_c = \langle F_{av} v \rangle = F_0 (\langle v \rangle + \kappa \langle v^2 \rangle) = F_0 \kappa \langle v^2 \rangle \quad \text{cooling rate}$$

$$\dot{E}_h = \frac{1}{2m} \frac{d}{dt} \langle P^2 \rangle = \dot{E}_{abs} + \dot{E}_{emit} \quad \text{sum of abs. and emiss. rate}$$

$$= \dot{E}_{abs}(1 + \xi) \approx \frac{1}{2m} (\hbar k)^2 \Gamma \rho_{ee}(v = 0)(1 + \xi) \quad \text{heating rate}$$

$$m\langle v^2 \rangle = k_B T = \frac{\hbar \Gamma}{8} (1 + \xi) \left[ \frac{(1+s)\Gamma}{2\Delta} + \frac{2\Delta}{\Gamma} \right] \quad \text{final energy}$$

$$T_{DL} = \frac{\hbar \Gamma \sqrt{1+s}}{4k_B} (1 + \xi) \quad \text{final temperature}$$

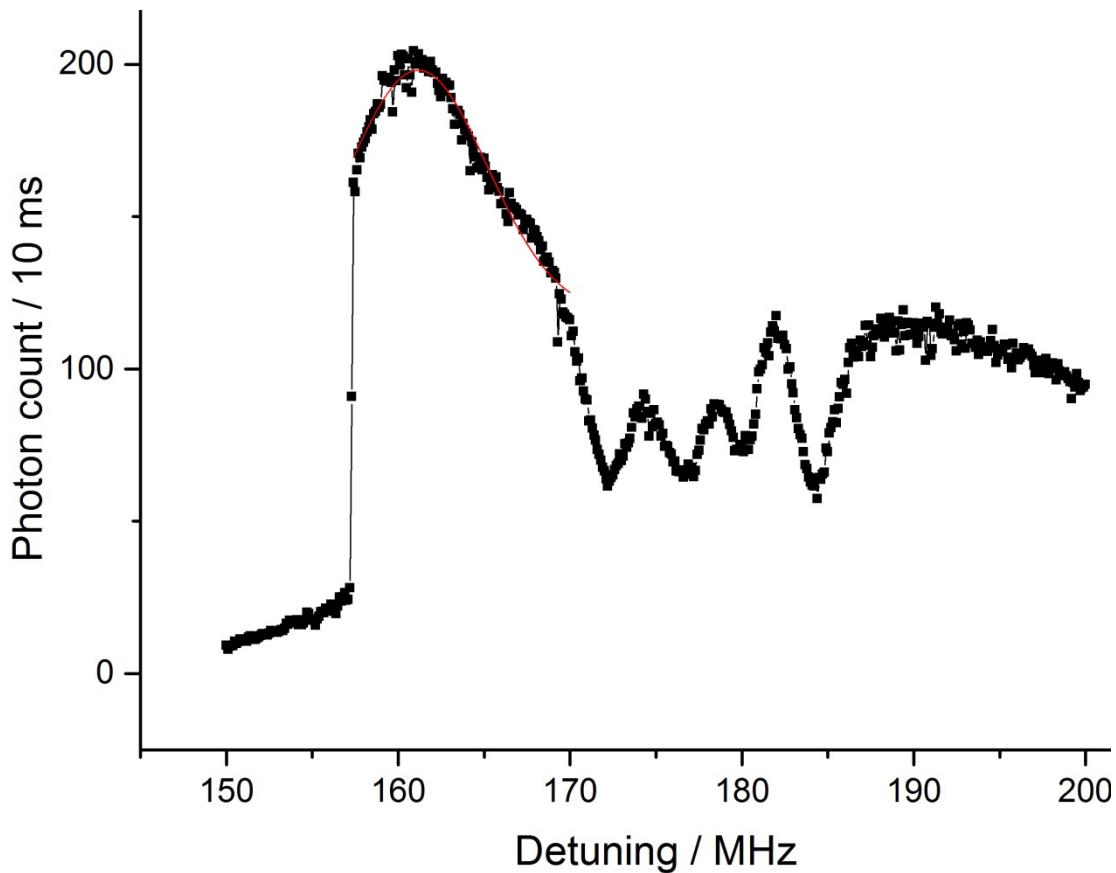
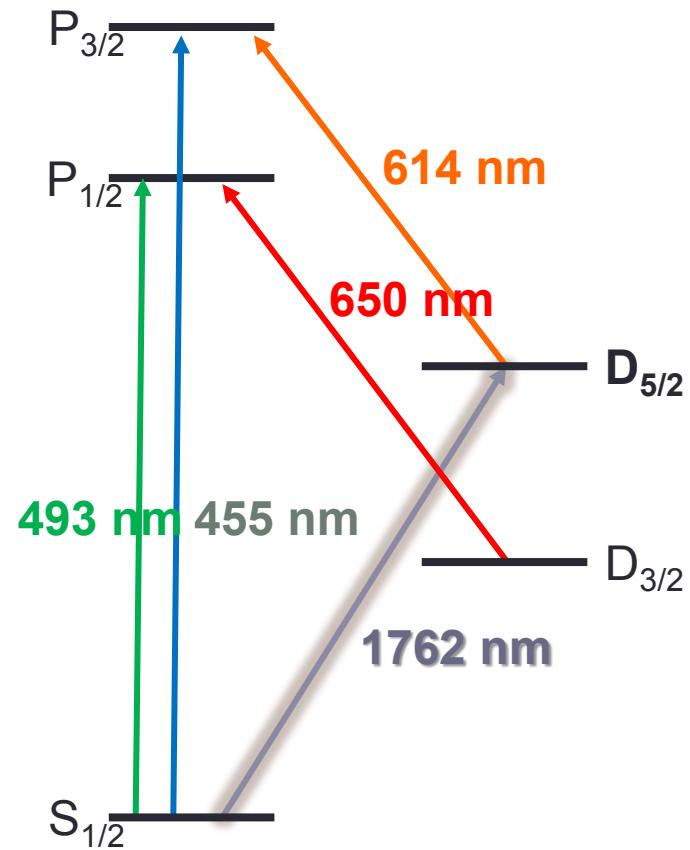
Best results

$$\Delta = \Gamma \sqrt{1 + \frac{s}{2}}$$

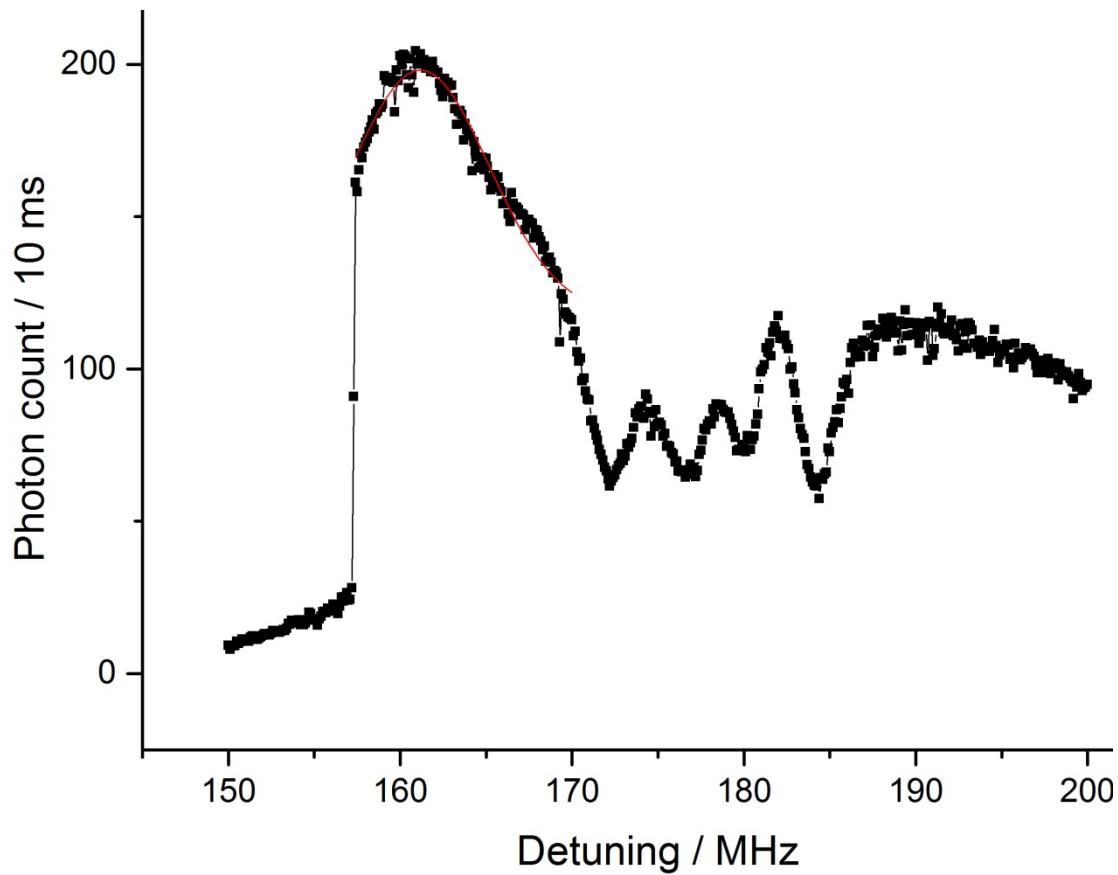
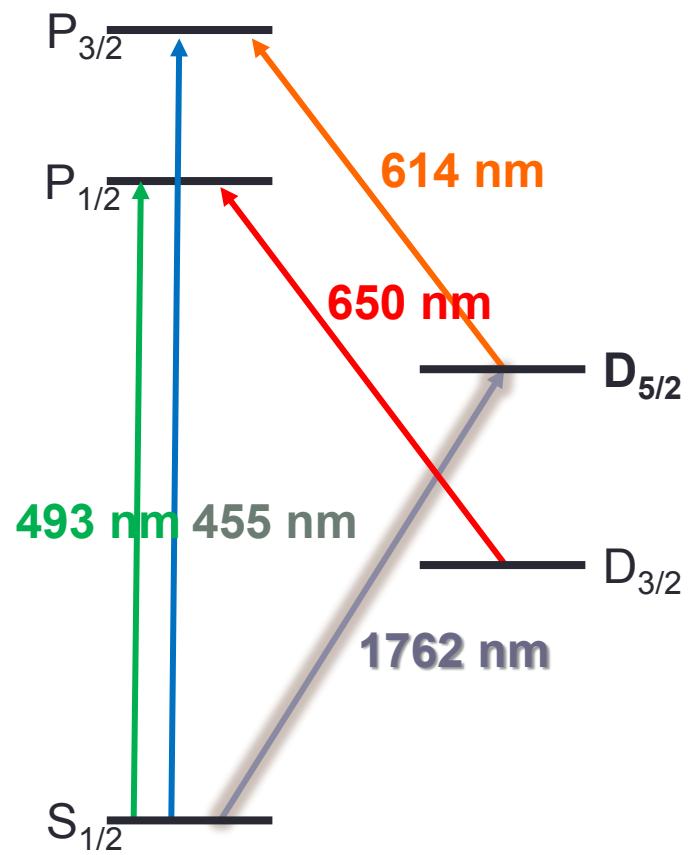
$$s = 2 \frac{|\Omega|^2}{\Gamma^2}$$

Required detuning  
saturation parameter

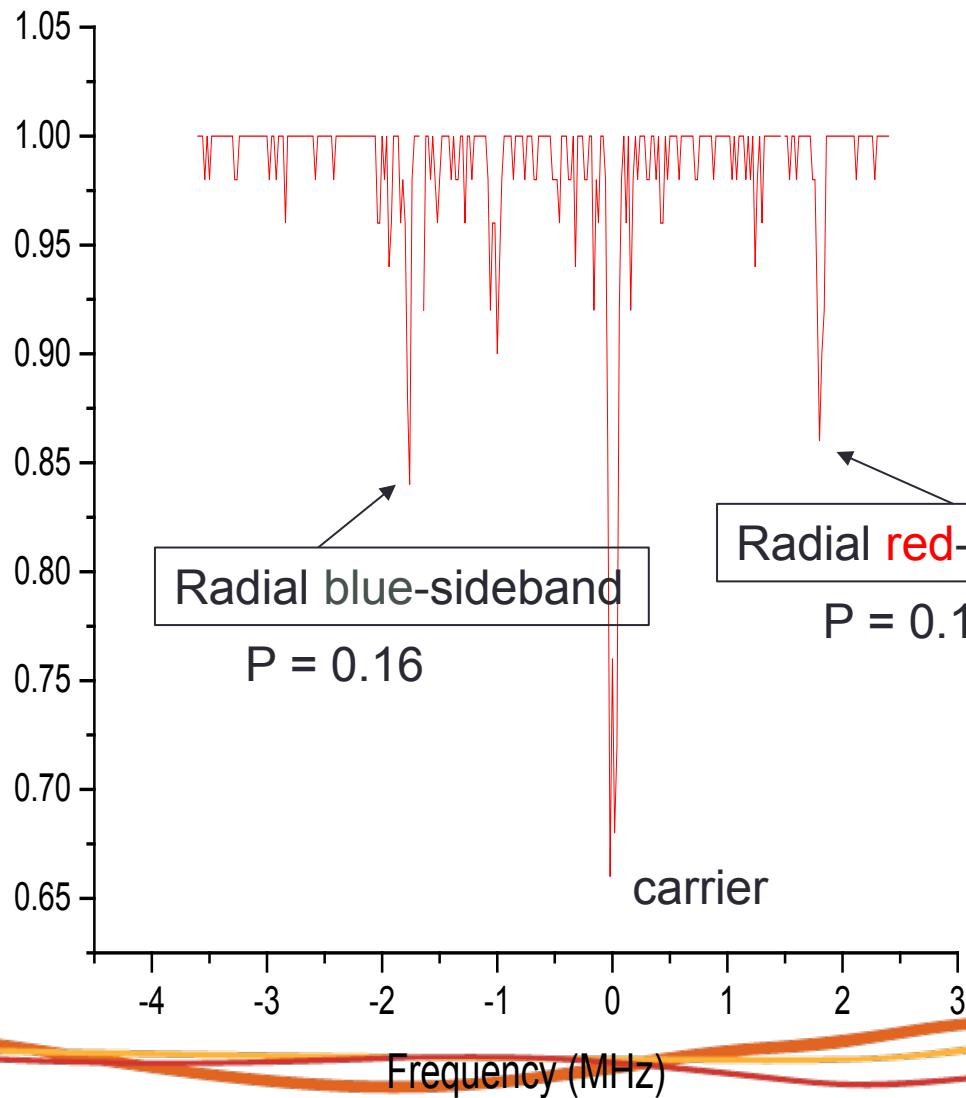
# Doppler cooling – profile



# Doppler cooling – profile



# Carrier and Side-band



Power of rf = 4 W  
(trap frequency  $\sim 1.8$  MHz)

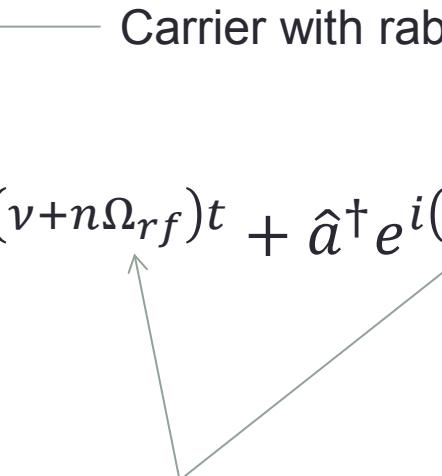
$$\frac{P_{\text{red}}}{P_{\text{blue}}} = \frac{n}{n+1} = \frac{0.14}{0.16}$$

$n = 7$

# Resolved sideband

Expanding in terms of  $\eta$  and keeping upto the second term

$$\begin{aligned}
 \hat{H}_{inter}^{LD}(t) = & \frac{\hbar}{2} \Omega [\hat{\sigma}_+ e^{-i\delta t} + h.c.] \quad \xleftarrow{\hspace{1cm}} \text{Carrier with rabi frequency } \Omega \\
 & + \frac{\hbar}{2} \Omega \left\{ \sum_{n=-\infty}^{\infty} i\eta C_{2n} \hat{\sigma}_+ e^{-i\delta t} \times \left[ \hat{a} e^{-i(\nu+n\Omega_{rf})t} + \hat{a}^\dagger e^{i(\nu+n\Omega_{rf})t} \right] \right. \\
 & \left. + h.c. \right\}
 \end{aligned}$$


  
 Sidebands  $\pm(\nu + n\Omega_{rf})$  with strength  $\eta C_{2n} \Omega$

Condition for validity of resolved sideband

$$\Omega_{rf} \ll \nu \ll \tilde{\Gamma}$$



# Sideband cooling to ground state

Adjust detuning  $\delta = \omega - \omega_a = -\nu$  ( $n = 0$ )

$$H_{inter}^{LD} = \frac{\hbar}{2} \Omega [\hat{\sigma}_+ e^{i\nu t} + \hat{\sigma}_- e^{-i\nu t} + i\eta (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger) + i\eta (\hat{\sigma}_+ \hat{a}^\dagger e^{i2\nu t} + \hat{\sigma}_- \hat{a} e^{-i2\nu t})]$$

Carrier shifted by  $\nu$

blue sideband at  $2\nu$

resonant sideband

Cooling rate:

$R_n = \text{excited state occupancy probability } (P_e(n))$   
 $\times \text{decay rate}$



# Sideband cooling to ground state

Cooling rate

$$R_n = \text{excited state occupancy probability } (P_e(n)) \\ \times \text{decay rate}$$

$$R_n = \tilde{\Gamma} P_e(n) = \tilde{\Gamma} \frac{(\eta\sqrt{n}\Omega)^2}{2(\eta\sqrt{n}\Omega)^2 + \tilde{\Gamma}^2}$$

- The rate is depended on n
- The rate vanishes as n=0 is approached
- The final motional state is a dark state
- Dominant contribution to heating comes from carrier and 1<sup>st</sup> blue sideband absorption



# Sideband cooling

Restricting to the first two motional states the rate equation (equilibrated heating and cooling):

$$\begin{aligned}\dot{P}_0 &= P_1 \frac{(\eta\Omega)^2}{\tilde{\Gamma}} - P_0 \left[ \left( \frac{\Omega}{2\nu} \right)^2 \tilde{\eta}^2 \tilde{\Gamma} + \left( \frac{\eta\Gamma}{4\nu} \right)^2 \tilde{\Gamma} \right] \\ \dot{P}_1 &= -\dot{P}_0\end{aligned}$$

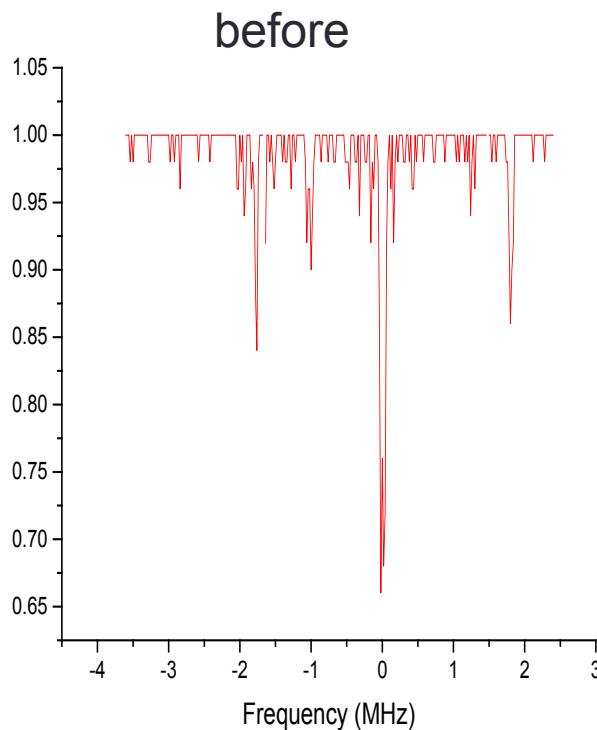
$P_i$  are probabilities to be in state  $|i\rangle$

In steady state  $\dot{P}_i = 0$

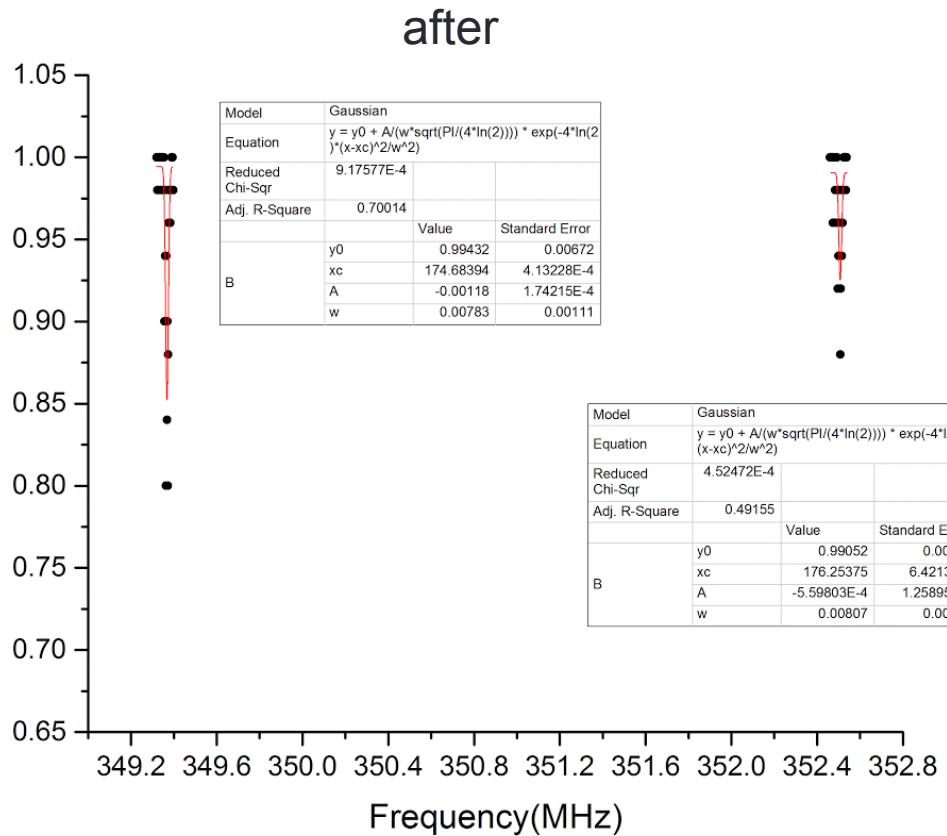
$$\bar{n} \approx P_1 \approx \left( \frac{\tilde{\Gamma}}{2\nu} \right)^2 \left[ \left( \frac{\tilde{\eta}}{\eta} \right)^2 + \frac{1}{4} \right]$$



# Sideband cooling



$$\bar{n} = 7$$



$$\bar{n} = 0.9$$



# Motional state population

$$|\psi(0)\rangle = |g\rangle \sum_{n=0}^{\infty} c_n |n\rangle \quad \text{Initial state}$$

$$P_g(t) = \langle \psi(t) | (|g\rangle\langle g| \otimes \hat{I}_m) |\psi(t)\rangle$$

probability to be in the ground state after excitation

$$P_g(t) = \frac{1}{2} [1 + \sum_{n=0}^{\infty} P_n \cos \Omega_{n,n+1} t] \quad \text{after blue sideband excitation}$$

$$P_n = |c_n|^2 \quad \text{probabilities to be in motional } n\text{-state}$$



# Motional state after cooling

1. Final state is a thermal state
2. Use  $P_e(t) = 1 - P_g(t)$
3. Find the probability ratio of red-to-blue sideband excitations

$$\begin{aligned}
 P_e^{RSB}(t) &= \sum_{m=1}^{\infty} \left( \frac{\bar{n}}{\bar{n}+1} \right)^m \sin^2 \Omega_{m,m-1} t \\
 &= \frac{\bar{n}}{(\bar{n}+1)} \sum_{m=0}^{\infty} \left( \frac{\bar{n}}{\bar{n}+1} \right)^m \sin^2 \Omega_{m+1,m} t & \Omega_{m+1,m} = \Omega_{m,m-1} \\
 &= \frac{\bar{n}}{\bar{n}+1} P_e^{BSB}(t)
 \end{aligned}$$

$$R = \frac{P_e^{RSB}}{P_e^{BSB}} = \frac{\bar{n}}{\bar{n} + 1}$$

